THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 9

§65) 1) Recall that $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ for $|z| < \infty$. Therefore,

$$z\cosh(z^2) = z\sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

§65) 2) a) Since $f^{(n)}(z) = e^z$ for any $n \in \mathbb{N}$, we have $f^{(n)}(1) = e$. Hence

$$e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

b) $e^z = e \times e^{z-1} = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$.

§65) 3) For $|z| < \sqrt{2}$,

$$\begin{aligned} f(z) &= \frac{z}{z^4+4} \\ &= \frac{z}{4}\cdot\frac{1}{1-(-z^4/4)} \\ &= \frac{z}{4}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n}}{4^{2n}} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+2}}z^{4n+1} \end{aligned}$$

§65) 4) Since $\cos z = -\sin\left(z - \frac{\pi}{2}\right)$ and $\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$, we have

$$\cos z = -\sin\left(z-\frac{\pi}{2}\right) = -\sum_{n=0}^{\infty}(-1)^n\frac{(z-\pi/2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z-\pi/2)^{2n+1}}{(2n+1)!}$$

§65) 9) Since $\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$, we have

$$f(z) = \sin(z^2) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+2}}{(2n+1)!}$$

Since we also have $f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$, by comparing the coefficients, we have

$$f^{(4n)}(0) = 0 \text{ and } f^{(2n+1)}(0) = 0$$

for $n = 0, 1, 2, \ldots$.

§65) 11) For $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} \cdot \frac{1}{1 - \frac{z}{4}} = \frac{1}{4z} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \frac{1}{4z} + \sum_{n=1}^{\infty} \left(\frac{z^{n-1}}{4^{n+1}}\right) = \frac{1}{4z} + \sum_{n=0}^{\infty} \left(\frac{z^n}{4^{n+2}}\right)$$

§68) 1) For $0 < |z| < \infty$, $f(z) = z^2 \sin\left(\frac{1}{z^2}\right) = z^2 \sum_{n=0}^{\infty} (-1)^n \frac{z^{-4n-2}}{(2n+1)!} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!} \cdot \frac{1}{z^{4n}}$.

§68) 3) For $1 < |z| < \infty$, $f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z^3} \cdot \frac{1}{1 + \frac{1}{z^2}} = \frac{1}{z^3} \sum_{n=0}^{\infty} \left(\frac{-1}{z^2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+3}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}}$

§68) 5) For $|z| < 1$, we have

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

For $|z| > 1$, we have

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

Similarly, for $|z| < 2$, we have

$$\frac{1}{z-2} = -\frac{1}{2} \frac{1}{1 - \frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

For $|z| > 2$, we have

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1 - \frac{2}{z}} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{2^n}{z^n} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

As a result,

- in D_1,

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n$$

- in D_2,

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}$$

- in D_3,

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n}$$